Modal properties of base isolated buildings with soil-structure interaction

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ABSTRACT: The influence of soil-structure interaction and the possible effects of building and foundation rocking are examined by investigating the modal properties of base isolated buildings idealized as plane frames. Simplifications in the analysis are also examined.

1 INTRODUCTION

Base isolation is a novel concept for the protection of buildings against severe earthquakes. The reduction in seismic forces on the building is achieved by inserting horizontally soft isolation elements between the superstructure and the foundation. The inelastic deformations, if any, are limited to the isolating elements and the structure remains in the elastic range. The additional damping provided by the isolators further reduces the resonant component of the response.

Base isolation has attracted much interest as testified by the extensive literature on the subject (ATC 1986, Blakeley et al. 1979, Kelly 1986). In the usual type of analysis, soil-structure interaction is neglected and only sliding of the structure is accounted for. The latter assumption is adequate for the common designs in which the isolators are very stiff vertically relative to their horizontal stiffness and the soil is stiff. However, the soil may not always be very stiff, particularly in rocking or the base isolated building may rest on piles as is the case of one building in Japan.

The objectives of this paper are to explore the effects of soil-structure interaction and rocking resulting from it and from vertical flexibility of the isolators on modal properties of buildings. The study is limited to modal properties because they are clearly indicative of the benefits that can be expected from the types of base isolation considered.

2 EQUATIONS OF MOTION AND THEIR SOLUTION

Consider a low rise building on isolators that, in turn, rests on individual footings as indicated in Figures 1 and 2. The governing equations of free damped vibration may be written in the standard general form

$$[m]\{\ddot{u}\}+[c]\{\dot{u}\}+[k]\{u\} = \{0\} \tag{1}$$

where $[m]$, $[k]$ and $[c]$ are mass, stiffness and damping matrices respectively and $\{u\}$, $\{\dot{u}\}$, $\{\ddot{u}\}$ are absolute vectors of displacement, velocity and acceleration respectively. For soil-structure interaction, the bases of the isolators are not fixed but have stiffness, mass and damping associated with their degrees-of-freedom.

If the building is idealized as a plane frame, there are three degrees-of-freedom per joint, two translations and a rotation. When the degrees-of-freedom are numbered sequentially from the top of the structure down to the base, the following forms of the above matrices occur

$$[m]=\begin{bmatrix}[m_s] & \\ & [m_f]\end{bmatrix};\quad [c]=\begin{bmatrix}[c_s] & \\ & [c_f]\end{bmatrix};$$
$$[k]=\begin{bmatrix}[k_s] & \\ & [k_f]\end{bmatrix} \tag{2}$$

where the subscript f refers to the foundation degrees-of-freedom and s refers to the superstructure (including the isolators). Stiffness and damping terms of the superstructure are associated with every

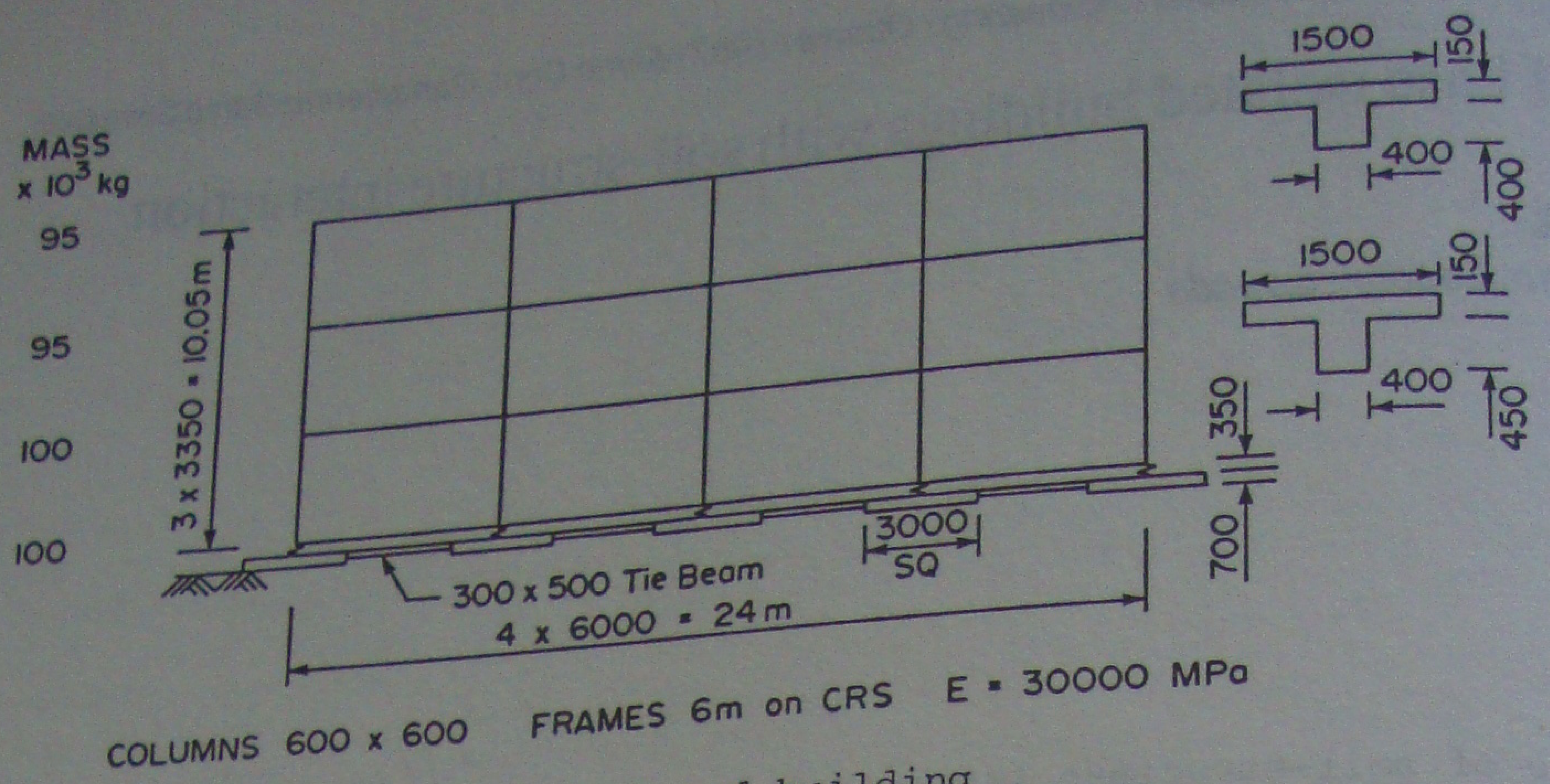


Figure 1 Three storey base isolated building

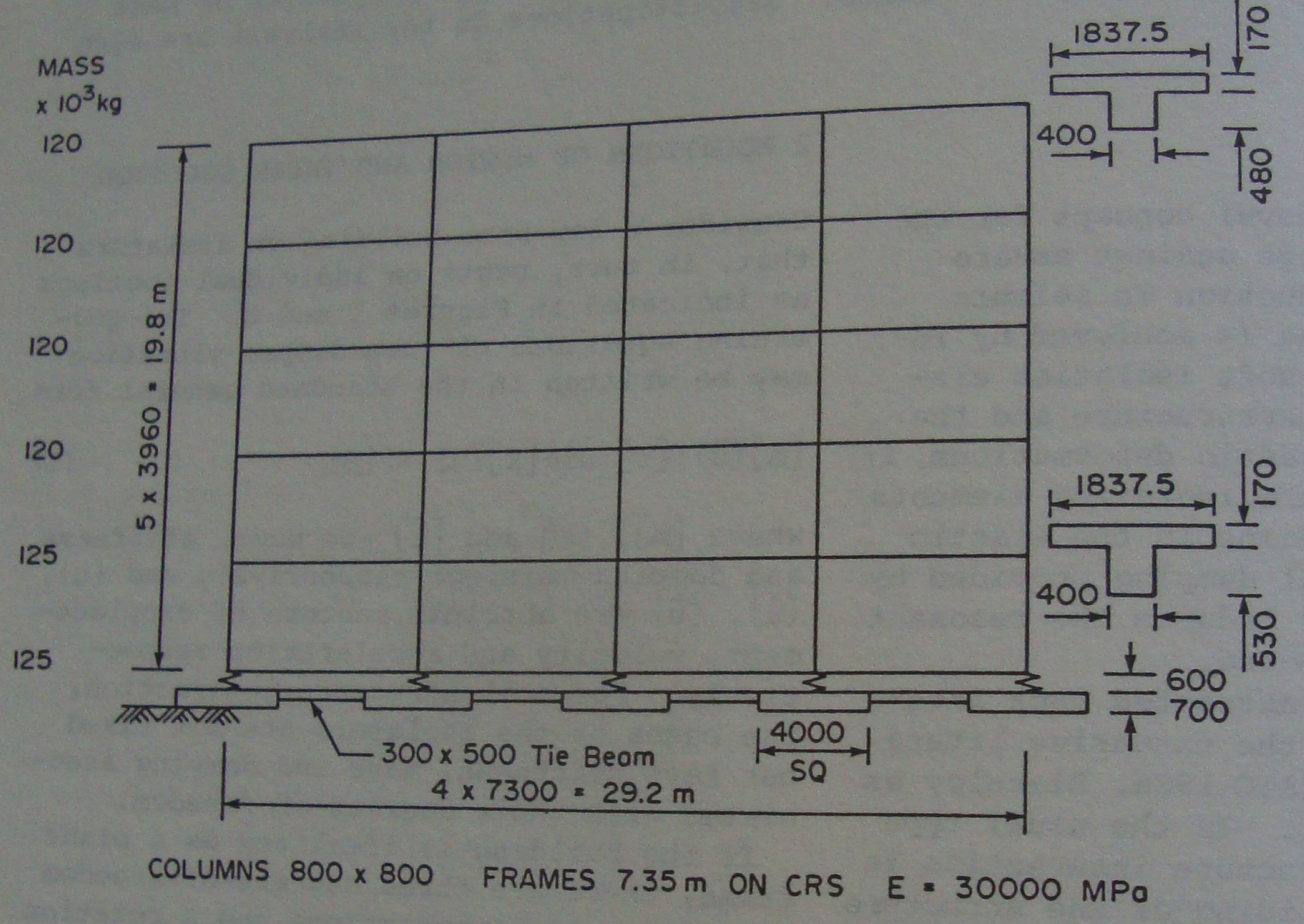


Figure 2 Five storey base isolated building

degree-of-freedom while the terms associated with the foundation are assembled (added) in their corresponding degrees-of-freedom. Thus, the structure and foundation stiffness and damping matrices overlap. A somewhat different and explicit form of the governing equations was formulated for base isolated buildings with soil-structure interaction resulting from sliding and rocking flexibilities of joint foundation mats by Novak and El Hifnawy (1986).

To save on computing costs, the matrices may be condensed dynamically with due regard to preserving the structure energies. Such a technique, described by Guyan (1965), is used herein. The horizontal translations are chosen as the masters.

When accounting for soil-structure interaction, the total damping matrix is non-proportional. To obtain modal damping accurately, non-classical or complex eigenvalue analysis can be employed. Such analysis yields the complex eigenvalues, μ_j, that occur in complex conjugate pairs. From these complex eigenvalues the damped circular natural frequency is obtained as

$$\omega_j' = \mathrm{Im}\,\mu_j \tag{3}$$

and the modal damping ratio as

$$\zeta_j = \frac{-\mathrm{Re}\,\mu_j}{|\mu_j|} \tag{4}$$

With each pair of complex eigenvalues, a pair of complex conjugate eigenvectors is associated. More details on the procedure can be found in Foss (1958), Novak and El Hifnawy (1983) and elsewhere.

3 BUILDINGS ANALYZED

The buildings analyzed are schematically depicted in Figures 1 and 2. The frames are reinforced concrete of monolithic construction. The effective slab width is taken as one-quarter the centre to centre distance of the frames. The gross cross-sections are used in computing the moment of inertia and the cross-sectional area.

The base isolators are capable of displacing in the horizontal and vertical directions and also in rotation. The ratio of the vertical to the horizontal stiffness is varied to reflect the type of isolator being used. The horizontal stiffness is so selected as to provide effective isolation of the building.

The footings are square and are assumed to rest on a viscoelastic halfspace with a shear wave velocity of 150 m/s.

The floors of the buildings are taken as rigid diaphragms. Consequently, all joints on each floor have the same horizontal translation. The same constraint applies to the footings that are connected by tie beams.

4 MASS, STIFFNESS AND DAMPING MATRICES

4.1 The masses

The mass of the floors and columns is lumped at the nodes on each floor. The floor rotational inertia is accounted for through vertical displacements of the joints. The isolators are assumed massless. The footings are treated as rigid bodies and thus possess both mass and mass moments of inertia.

4.2 The structure

The stiffnesses of the beams and columns are computed from the usual stiffness matrix for elastic plane frame members, matrix $[K_1]$. The columns are axially extensible.

Hysteretic behaviour of the members is modelled by a complex elastic modulus

$$E^* = E + iE' = E(1+i2\beta) \tag{5}$$

where $i = \sqrt{-1}$, E=real modulus and β is the material damping ratio assumed to be frequency independent and defined as

$$\beta = \frac{E'}{2E} = \frac{\Delta W}{4\pi W} \tag{6}$$

in which ΔW is the area bounded by the hysteretic loop and W is the strain energy. To account for structural damping, the complex stiffness matrix is formulated, using the complex elastic modulus, as $[K] = [K_1] + i[K_2]$, in which the imaginary component is $[K_2] = 2\beta[K_1]$. The matrix of equivalent viscous damping of structural members is defined as

$$[c_m] = \frac{[K_2]}{\omega_1} = \frac{2\beta}{\omega_1}[K_1] \tag{7}$$

where ω_1 is the first modal frequency. (An adjustment for higher modes is described later). For the reinforced concrete members, the material damping ratio, β, is taken as 1%.

4.3 The isolators

The force-displacement relationship of the base isolators is assumed to be governed by a bilinear hysteretic loop, as in Figure 3. The post yielding stiffness, k_2, is taken as 1.0 W per metre with W being the weight of the building. This stiffness represents the summation of the horizontal stiffnesses of the isolators. The initial stiffness, k_1, is taken as $5k_2$ and the yield force, P_y, as 0.05 W. Two levels of the displacement u_o are considered, termed small strain and large strain. For the first level of displacement, small strain, the isolators remain in the elastic range and their horizontal stiffness is equal to the initial stiffness, k_1; this state is for displacements $\leq$ 10 mm. For the second level of displacement, large strain, the isolators undergo large displacements for which a nominal value of 300 mm is adopted.

A vertical stiffness of 200 times the horizontal stiffness is chosen for the case of no rocking. In addition, the buildings are analyzed with a much smaller vertical stiffness of the isolators equal to twice the horizontal stiffness; this case involves building rocking as well as sliding.

The area of the hysteretic loop is a measure of the energy dissipated in the isolators. The hysteretic loop indicated in Figure 3 implies nonlinearity. However, the complex eigenvalue approach, employed in the analysis of the whole system, presumes linearity and viscous damping. One simple approximate way of overcoming this difficulty is to define the equivalent linear stiffness through the backbone curve of the hysteretic loop (a straight line in this case), and to derive the equivalent viscous damping ratio from the area of the hysteretic loop. Then, the equivalent linear stiffness becomes

$$k_e = P_o/u_o \tag{8}$$

where P_o is the force on the isolator and u_o is its displacement.

The material damping ratio of the base isolators, established using Equation 6, becomes, as shown already by Watanabe and Tochigi (1985),

$$\beta = \frac{2}{\pi}\,\frac{1-\varepsilon}{1+\varepsilon(x_o-1)}\,\frac{(x_o-1)}{x_o} \tag{9}$$

where $\varepsilon = k_2/k_1$ is the ratio of the post yielding stiffness to the initial stiffness and $x_o = u_o/u_y$ is the ratio of the displacement u_o to the yielding displacement u_y (Figure 3). For small strain, the damping ratio is taken as 1% while for large strain

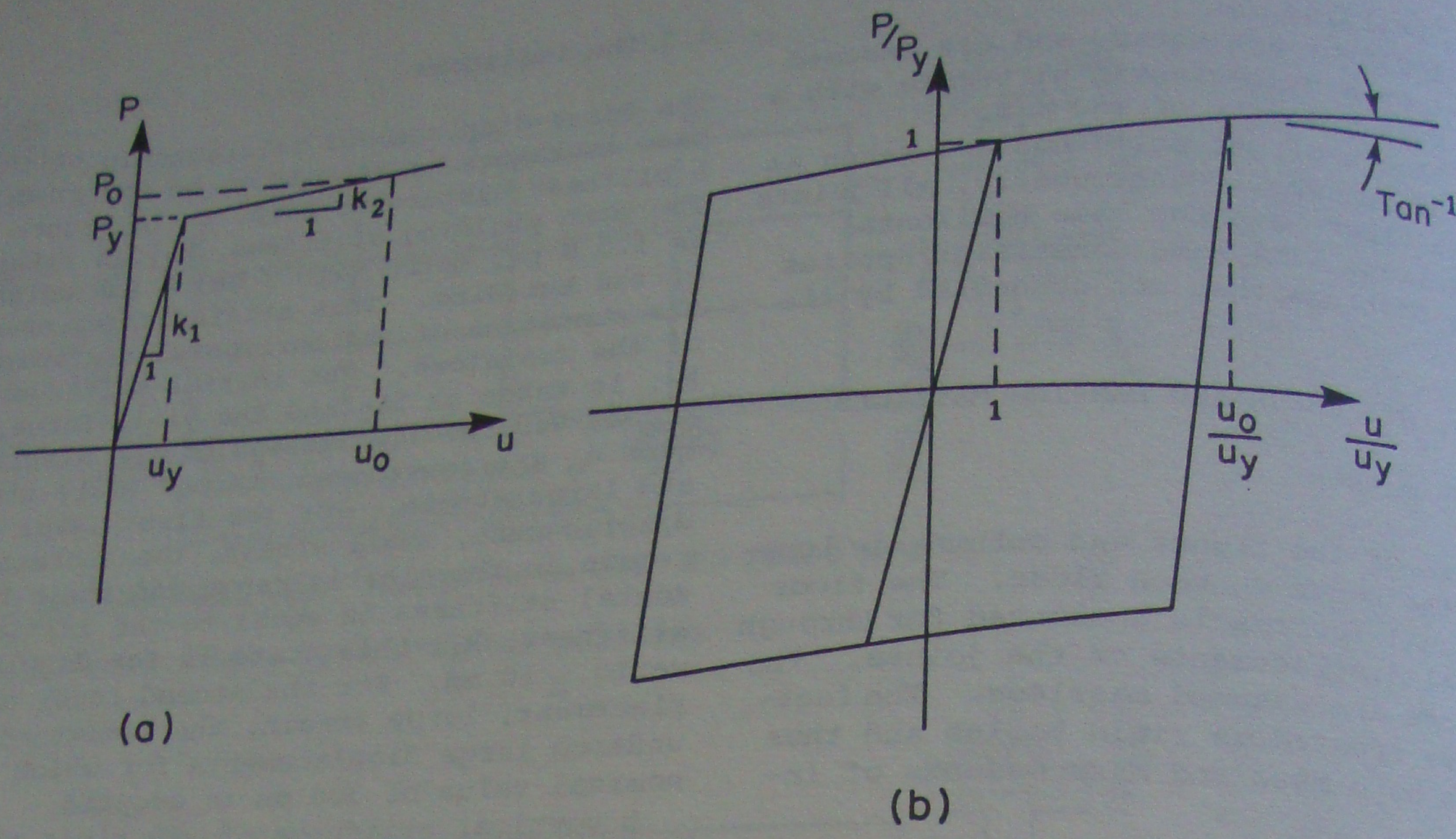


Figure 3 (a) Force-displacement relationship of base isolators.
(b) Normalized bilinear hysteretic loop

the computed damping ratio turns out to be 7.2%. The damping matrix is established from Equation 7.

4.4 The foundations

For impedance functions of rectangular, rigid footings, a number of formulae, charts and tables are available. Pais and Kausel (1985) used regression analysis to fit polynomial series to some of these data. Their expressions for horizontal translation, vertical translation and rocking were used in this study. The fundamental frequency of the building, ω_1, was used in computing the impedance functions. Material damping was included by means of the correspondence principle. The footing damping and stiffness matrices thereby obtained were transformed to the centre of mass of the foundation.

The mass, stiffness and damping matrices were assembled in global coordinates using standard algorithms. The horizontal translations at each floor and the foundation level were selected as the master degrees-of-freedom and the others were condensed. For a building with N storeys, the number of masters is N, N+1 and N+2 for the case of a fixed base, base isolation and base isolation with soil-structure interaction, respectively.

The eigenvalues and eigenvectors were computed using the EISPACK's RGG subroutine. The natural frequencies and modal damping ratio were computed from Equations 3 and 4. Since the damping matrices were established using the fundamental natural frequency, the damping in the higher modes was adjusted by the factor ω_1/ω_j giving

$$\zeta_j = \zeta_j' \frac{\omega_1}{\omega_j} \qquad (10)$$

where ζ_j' was the damping ratio based on the frequency ω_1.

5 TYPICAL RESULTS

The frames shown in Figures 1 and 2 as well as two other frames, half the width (i.e. with two bays), were analyzed for the following base conditions: a) fixed base, b) frames on base isolators with no rocking (isolated), c) frames on isolators with rocking, and d) frames on isolators with no rocking and soil-structure interaction included (isolated + soil). For the three and five storey buildings, the results are presented in Tables 1 and 2. The following observations emerge:

Base isolation shifts the fundamental frequency of the building away from the high energy portion of the earthquake spectrum; this coupled with the beneficial increase in damping can substantially reduce the

Table 1. Damped natural frequencies (Hz) and damping ratios (%) of base isolated *three storey buildings* (Isolation: $k_h = 5$ W/m, $\beta = 1\%$ for small strain or $k_h = 1.13$ W/m, $\beta = 7.2\%$ for large strain; $k_v = 200\ k_h$ for isolated and $k_v = 2k_h$ for rocking; footing area 9 m^2; soil: $V_s = 150$ m/s, $\beta = 5\%$, $\nu = 0.25$, $\rho = 1800$ kg/m^3).

		Mode	1		2		3		4		5	
		Base	Hz	%	Hz	%	Hz	%	Hz	%	Hz	%
Two Bays		Fixed	3.50	1.0	11.36	1.0	20.7	1.0	-	-	-	-
	Small Strain	Isolated	1.07	1.0	5.28	1.0	12.1	1.0	20.1	1.0	-	-
		Iso&Soil	1.06	1.1	4.97	1.6	12.1	1.0	19.8	1.0	23.8	>100
		Rocking	0.74	1.0	1.77	1.0	11.9	1.0	17.5	1.0	-	-
	Large Strain	Isolated	0.52	7.1	4.23	2.9	11.8	1.1	18.2	1.1	-	-
		Iso&Soil	0.52	7.1	3.98	3.3	11.8	1.1	18.1	1.1	23.9	>100
		Rocking	0.36	7.1	0.82	7.2	11.3	1.1	15.5	1.2	-	-
Four Bays		Fixed	3.38	1.0	10.73	1.0	19.1	1.0	-	-	-	-
	Small Strain	Isolated	1.07	1.0	5.26	1.0	11.6	1.0	18.7	1.0	-	-
		Iso&Soil	1.06	1.1	5.11	1.3	11.6	1.0	18.6	1.0	24.0	>100
		Rocking	0.91	1.0	1.84	1.0	11.4	1.0	16.2	1.0	-	-
	Large Strain	Isolated	0.52	7.1	4.65	1.9	11.3	1.0	17.4	1.1	-	-
		Iso&Soil	0.52	7.1	4.43	2.1	11.3	1.0	17.2	1.1	22.7	>100
		Rocking	0.44	7.0	0.81	7.0	11.1	1.0	15.1	1.1	-	-

Table 2. Damped natural frequencies (Hz) and damping ratios (%) of base isolated *five storey buildings* (Isolation: $k_h = 5$ W/m, $\beta = 1\%$ for small strain or $k_h = 1.13$ W/m, $\beta = 7.2\%$ for large strain; $k_v = 200\ k_h$ for isolated and $k_v = 2\ k_h$ for rocking; footing area 16 m^2; soil: $V_s = 150$ m/s, $\beta = 5\%$, $\nu = 0.25$, $\rho = 1800$ kg/m^3)

		Mode	1		2		3		4		7	
		Base	Hz	%	Hz	%	Hz	%	Hz	%	Hz	%
Two Bays		Fixed	2.06	1.0	6.81	1.0	13.16	1.0	21.00	1.0	-	-
	Small Strain	Isolated	0.99	1.0	3.55	1.0	7.78	1.0	13.19	1.0	-	-
		Iso&Soil	0.97	1.3	3.36	1.7	7.77	1.0	12.91	1.1	20.87	>100
		Rocking	0.55	1.0	1.86	1.0	7.69	1.0	10.58	1.0	-	-
	Large Strain	Isolated	0.51	6.9	2.88	2.9	7.60	1.1	12.27	1.2	-	-
		Iso&Soil	0.51	6.9	2.72	3.3	7.60	1.1	12.04	1.2	20.79	>100
		Rocking	0.27	7.1	0.86	7.3	7.32	1.1	9.7	1.1	-	-
Four Bays		Fixed	2.02	1.0	6.56	1.0	12.42	1.0	19.47	1.0	-	-
	Small Strain	Isolated	0.98	1.0	3.56	1.0	7.55	1.0	12.61	1.0	-	-
		Iso&Soil	0.97	1.2	3.47	1.4	7.55	1.0	12.46	1.1	20.92	>100
		Rocking	0.74	1.0	1.89	1.0	7.49	1.0	9.64	1.0	-	-
	Large Strain	Isolated	0.51	6.9	3.10	2.0	7.38	1.1	11.98	1.1	-	-
		Iso&Soil	0.51	6.9	3.00	2.3	7.38	1.1	11.81	1.1	20.82	>100
		Rocking	0.37	6.9	0.85	7.1	7.27	1.1	8.98	1.0	-	-

seismic loading of the building. However, the reduction in the fundamental frequency brings the building into the high energy portion of the wind spectrum making the base isolated building sensitive to dynamic wind loading and increasing the total wind load. These trends are schematically depicted in Figure 4.

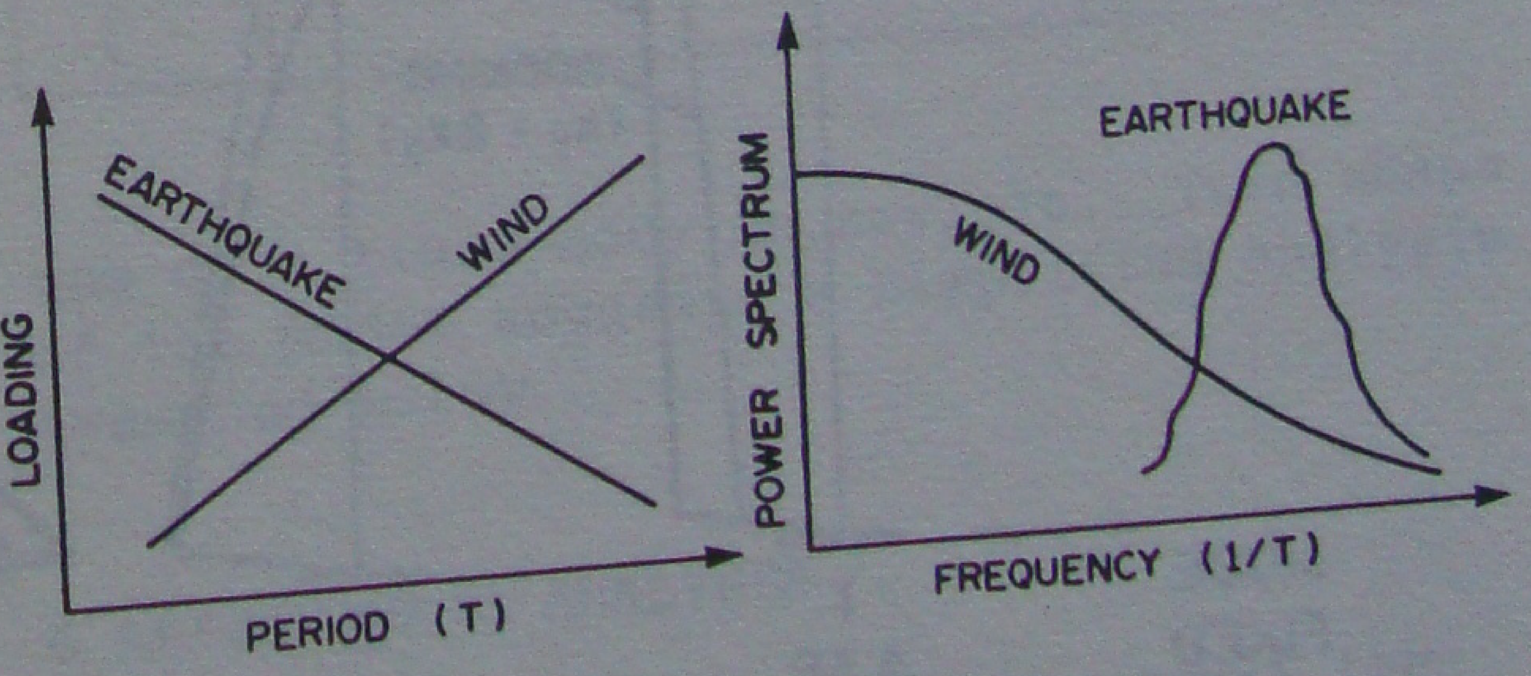


Figure 4. Schematic of earthquake and wind loading vs fundamental period with corresponding power spectra.

The rocking isolators are more effective in reducing the fixed base frequencies in modes 1 and 2 and provide more damping in modes 1 and 2, particularly for large strains and the second mode.

The three-storey frames benefit more from the rocking isolators in the second mode for large strains than the five storey frames. In the third mode, there are only small differences between the various base conditions.

Soil-structure interaction is negligible for the buildings being examined herein but can have a significant effect for other conditions, i.e. for softer soils, piles, stiffer isolators or taller buildings.

The *mode shapes* of the three storey building are shown for four bays in Figure 5, and for two bays in Figure 6. For no rocking, the buildings slide on the isolators in the first mode with the mode shape being almost unity throughout. For rocking, the buildings rock as well as slide on the isolators.

The damping in the first mode is equal to the damping assigned to the isolator. This is also true in the second mode for the rocking isolators. The second mode is influenced quite significantly by rocking. The mode shapes display a linear variation of displacements for rocking but a nonlinear one for the case of no rocking. The shape of the second mode is favourable because it results in a small participation factor. This is desirable because the higher mode frequencies are shifted closer to the peak in the earthquake spectrum. In the highest mode, i.e. the fifth in Figure 5, the foundation vibrates and the structure is almost motionless. This mode is heavily damped due to foundation damping.

The second mode shapes normalized to unity at the top of the building are shown in Figure 7. The modal coordinate corresponding to horizontal translation at the level of the isolator is much larger for rocking than in the absence of rocking. This factor, together with the larger vertical deformation of the isolators, combine to more effectively reduce the frequencies and provide higher damping in the second mode. To examine the effect of the vertical flexibility of the isolators in more detail, their damping ratio was increased from 7.2% to 30% in vertical translation. For a three storey building, the results for the first and second modes are displayed in Table 3. (For a five storey building, the results are similar.) For the isolated case, the first mode damping is approximately equal to the damping of the isolator in sliding. However, the second mode damping is increased which indicates that rocking is influential even in the isolated case (i.e. with vertically stiff isolators). For the rocking isolators, the damping in the first mode is significantly greater than in the isolated case. For the second mode, the more significant increase in damping occurs with the four bay, wider building. These results indicate that the use of discrete vertical dampers, in buildings on rocking isolators, can be quite beneficial.

Figure 8 shows the first two modes for the five storey building on rocking isolators for small strain evaluated using the

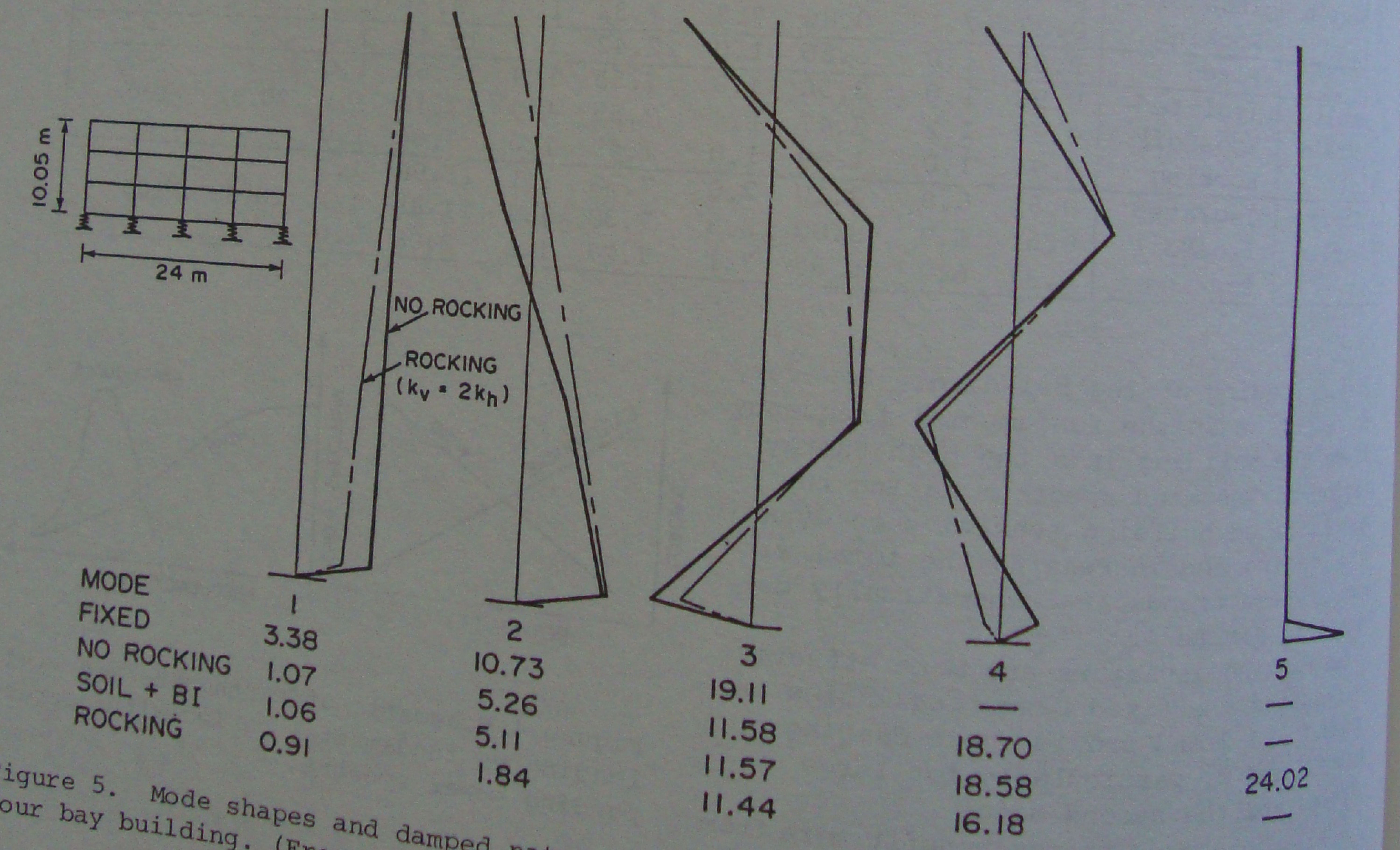
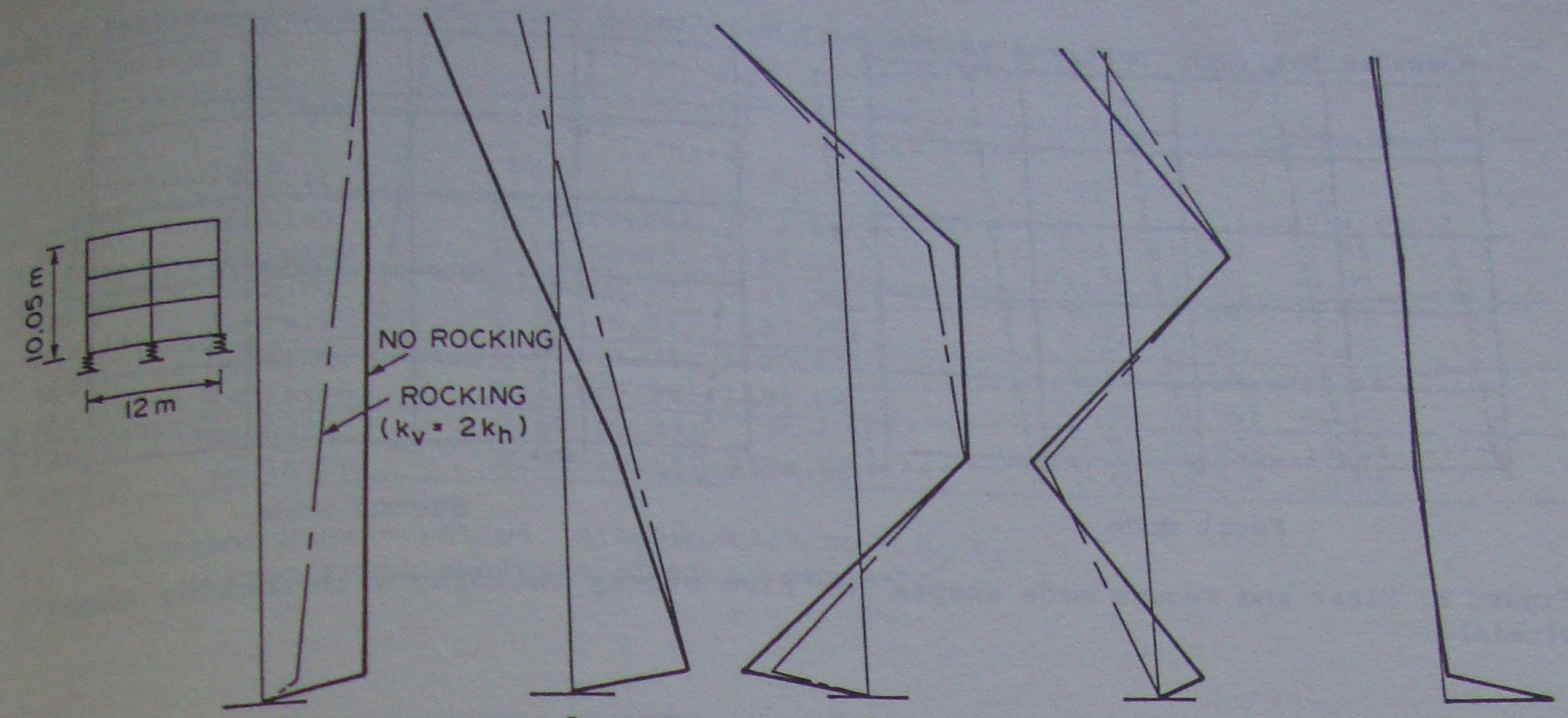


Figure 5. Mode shapes and damped natural frequencies of base isolated three storey-four bay building. (Frequency in Hz.)



MODE	1	2	3	4	5
FIXED	3.50	11.36	20.73	—	—
NO ROCKING	1.07	5.28	12.12	20.06	—
SOIL + BI	1.06	4.97	12.11	19.83	23.82
ROCKING	0.74	1.77	11.93	17.46	—

Figure 6. Mode shapes and damped natural frequencies (Hz) of base isolated three storey-two bay building

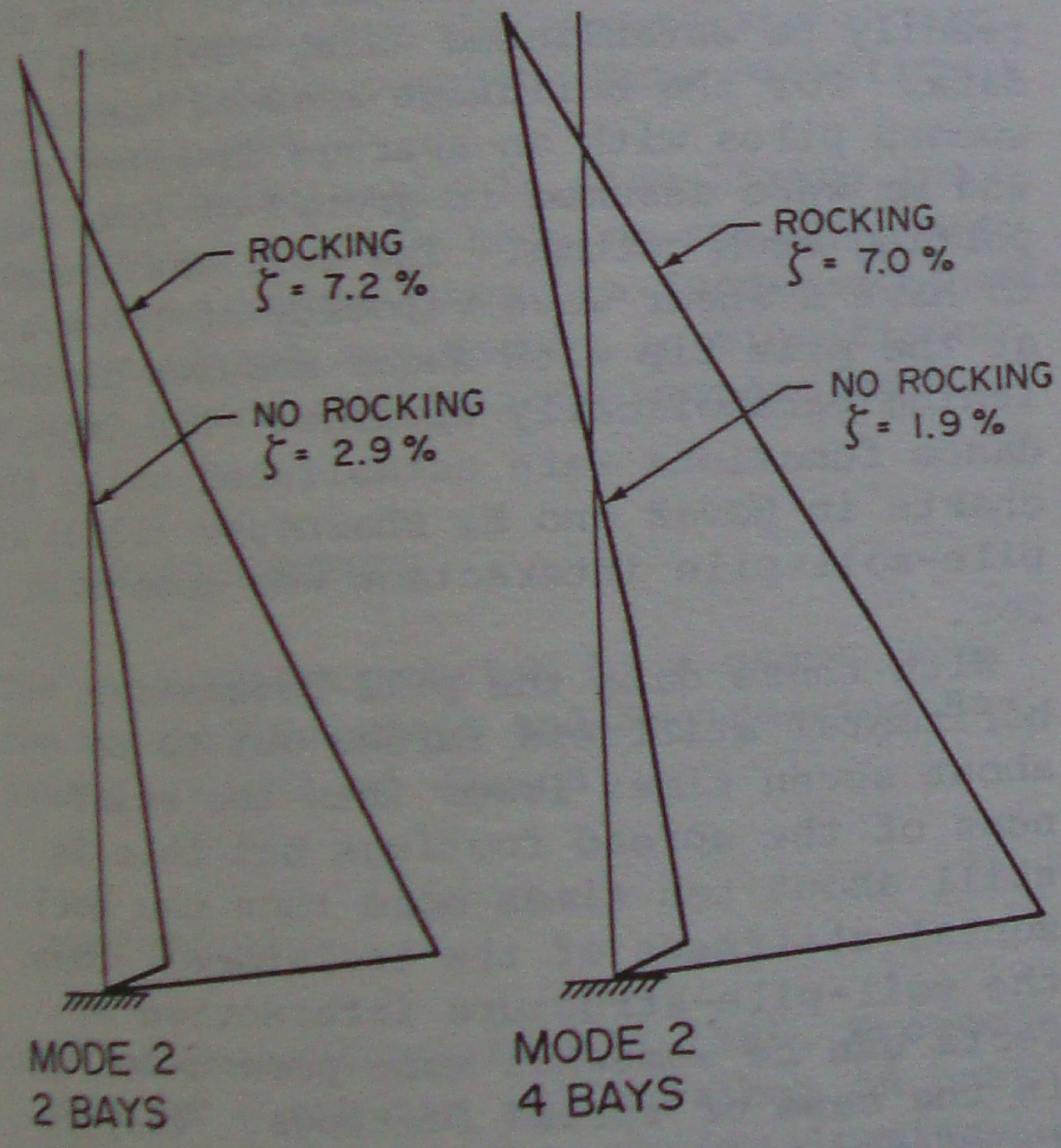


Figure 7. Influence of rocking on the second modal properties of three storey building.

PAFEC finite element program. Rotations and translations were included in the analysis. Some flexing of the superstructure occurs but the general trend seems to be one of a rigid body on springs. This shape of the modes suggests a simplified analysis examined below.

Table 3. Damped natural frequencies (Hz) and damping ratios (%) of base isolated three storey buildings with the vertical damping ratio increased from 7.2% to 30% (large strain; isolated: $k_v = 200\ k_h$, rocking: $k_v = 2\ k_h$)

		Mode	1		2	
		Base	Hz	%	Hz	%
Bays	Two	Isolated	0.52	7.2	2.79	14.7
		Rocking	0.36	21.7	0.76	16.5
	Four	Isolated	0.52	7.1	4.29	5.3
		Rocking	0.45	15.0	0.74	22.1

6 APPROXIMATE ANALYSES

In order to obtain the first natural frequency, the buildings could be modelled as a rigid body sliding on the isolators. The circular frequency is given by

$$\omega_1 = \sqrt{k/M} = \sqrt{aW/M} = \sqrt{ag} \qquad (11)$$

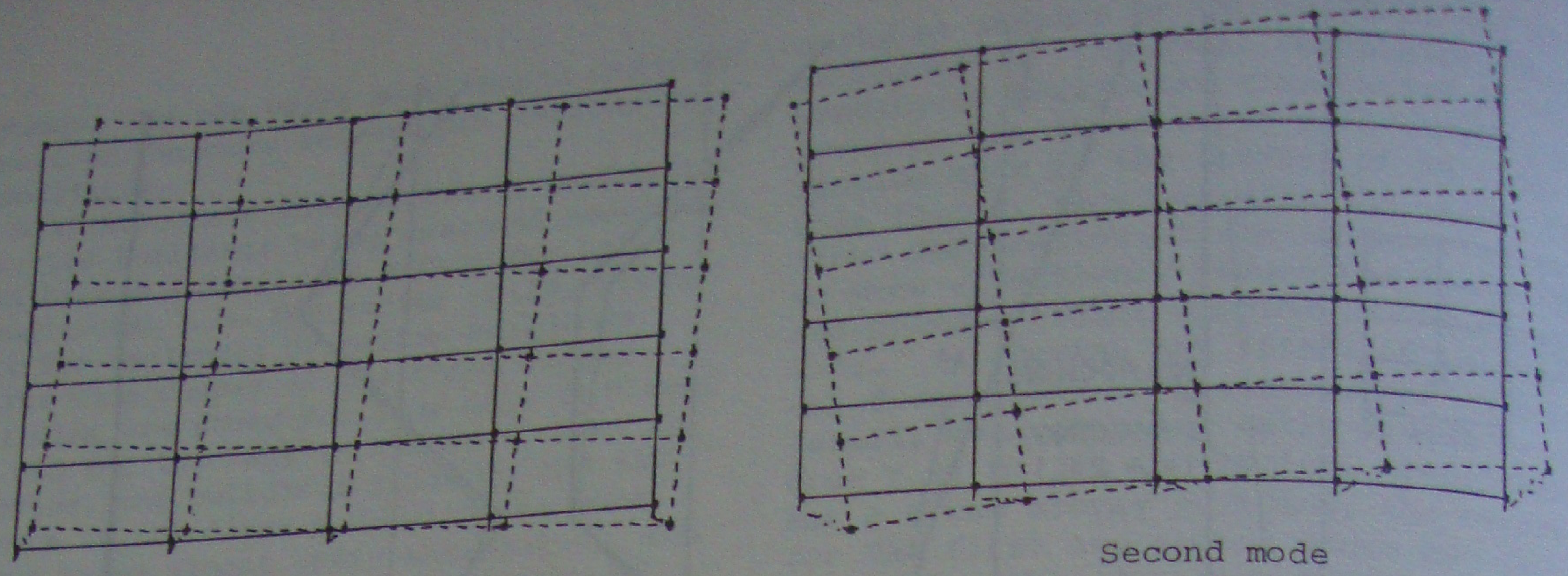


Figure 8. First and second mode shapes for five storey building with rocking (small strain).

where k = aW is the horizontal stiffness of the isolators, W the weight of the building and M its mass; a is a coefficient and g is the acceleration due to gravity (9.81 m/s^2).

The cyclic frequency is then

$$f_1 = 0.498 \sqrt{a} \quad (12)$$

For small strain, a=5 while for large strain a=1.13. Therefore, f_1=1.11 Hz for small strain and f_1=0.53 Hz for large strain. These values compare well with 1.07 Hz and 0.52 Hz for the three storey building and 0.99 Hz and 0.51 Hz for the five storey building.

The buildings could be analyzed approximately as a rigid block on springs featuring two degrees-of-freedom at the centre of mass, i.e. a horizontal translation and a rotation. (Such analysis was employed by Kelly and Pan, 1984.) Incorporating both mass and mass moment of inertia, this analysis yields the first and second mode shapes and frequencies given in Table 4. The modal damping is equal to the damping ratio assigned to the isolators.

The fundamental frequencies compare well with those obtained previously. For the rocking isolators, the second natural frequencies are fairly close. For the isolated condition, the second natural frequencies differ substantially from those of the plane frame. The differences can be attributed to the flexing and shearing of the frame members. The modal damping ratios compare well except in the second mode for the isolated condition where the rigid body damping is substantially higher than the accurate value. For the second mode and the isolated condition, the building rocks about a point quite close to its centre of mass while for the rocking condition, the centre of rotation is located above the centre of mass.

7 PILE FOUNDATIONS

In some cases, piles may be used in conjunction with base isolators. Piles are quite flexible in the horizontal direction and if they are friction piles, even in rocking. Impedance functions of piles can readily be established using published data. For the buildings assumed herein, wooden piles with an average diameter of 254 mm were assumed in groups of four supporting each column. The soil was assumed to have a shear wave velocity of 150 m/s at the pile tip with shear modulus diminishing parabolically upwards. The impedance functions were established using the charts in Novak and El Sharnouby (1983) and pile-soil-pile interaction was accounted for.

With these data the pile foundation horizontal stiffness turned out to be about seven times lower than the stiffness of the spread footings but this is still about ten times more than the horizontal stiffness of the isolators. Thus, the soil-pile-structure interaction effects can be somewhat more pronounced than in the case of spread footings. They reduced the first mode damping to 6.4% from 6.9%.

8 CONCLUSIONS

The theoretical study of the modal properties of base isolated buildings suggests the following conclusions:

Base isolation reduces seismic loading but also makes the building much more wind sensitive.

Rocking in the isolators more effectively

Table 4. Frequencies and vibration modes* for five storey building idealized as rigid body on springs

	Mode	1		2	
	Base	Hz	a_1	Hz	a_2
Two Bays, Small strain	Rocking	0.59 (0.55)	13.98	2.17 (1.86)	-3.61
	Isolated	1.10 (0.99)	533.26	11.57 (3.55)	-0.09
Two Bays, Large strain	Rocking	0.28 (0.27)	13.98	1.03 (0.86)	-3.61
	Isolated	0.52 (0.51)	533.26	5.50 (2.88)	-0.09
Four Bays, Small strain	Rocking	0.80 (0.74)	20.67	1.93 (1.89)	-5.02
	Isolated	1.11 (0.98)	1584.08	13.88 (3.56)	-0.07
Four Bays, Large strain	Rocking	0.38 (0.37)	20.67	0.92 (0.85)	-5.02
	Isolated	0.53 (0.51)	1584.08	6.60 (3.10)	-0.07

* a_j = vibration mode = ratio: sliding/rocking = u_j/ψ_j
Frequencies from frame analysis shown in brackets.

reduces the fixed base frequencies and provides a significant increase in the second mode damping. This should assist in reducing the resonant response especially in the second mode where the frequency is brought closer to the high energy of the earthquake spectrum.

Lower buildings benefit more from the rocking isolators.

The effect of soil-structure interaction on the modal properties of the base isolated buildings with the parameters chosen is negligible but may be more significant in other cases.

A simplified analysis, idealizing the building as a rigid body on springs, approximately establishes the fundamental mode properties but may result in significant errors in the second mode properties.

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